Q: Can a sequence converge if it is not monotone & bounded?

A: Yes, e.g. $s_n = \frac{(-1)^n}{n}$

By the squeeze thm, since $\left(\frac{1}{n}\right)$ and $\left(\frac{-1}{n}\right)$ converge to 0 &

$$-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n} \quad \forall n \in \mathbb{N},$$

the sequence $\left(\frac{(-1)^n}{n}\right)$ converges to 0 also.

Partial Converse to MCT:

Every convergent sequence is bounded.

Pf: Suppose $\lim a_n = L$.

Let $\varepsilon = 1$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n \ge N$, $|a_n - L| < \underset{\substack{\shortparallel \\ 1}}{\varepsilon}$. Then, by the triangle ineq.;

$$|a_n| - |L| \le |a_n - L| < 1$$

$$\Rightarrow \quad |a_n| < 1 + |L| \quad \text{for all } n \ge N.$$

Then $|a_n| \le \underbrace{\max\{1+|L|, |a_1|, |a_2|, \dots, |a_{N-1}|\}}_{\in \mathbb{R}}$. $\forall n \in \mathbb{N}$

So $|a_n|$ is bounded - $\exists M$ = that
s.t. $|a_n| \le M \quad \forall n \in \mathbb{N}$. ☒

Cor: If (b_n) is a sequence that is not bounded, then (b_n) diverges.

(contrapositive)

Important point: It is possible for a divergent sequence to be bounded (like $((-1)^n)$.)

Continuing our example

$$\left(x_1 = \sqrt{4} = 2, \quad x_{n+1} = \sqrt{4 + x_n} \quad \forall n \in \mathbb{N}. \right)$$

Last time - we showed that this sequence (x_n) converges, because it is increasing and is bounded ($1 \le x_n \le 10$.) — Thus we can use the MCT.

Now we can find the limit.

Lemma. If (a_n) converges to L, then (a_{n+1}) converges to L.

Pf: $\forall \varepsilon > 0$, because $(a_n) \to L$, $\exists N$ s.t. $\forall n \geq N$, $|a_n - L| < \varepsilon$.
but $n+1 \geq N$ also, so
$|a_{n+1} - L| < \varepsilon$. $\blacksquare$

Back to our sequence.

$$x_1 = 2$$
$$x_{n+1} = \sqrt{4 + x_n} \text{ for } n \geq 1.$$

We know $\exists L \in \mathbb{R}$ s.t. $x_n \to L$.

By the lemma,

$$\lim x_{n+1} = \underline{\lim x_n = L}.$$

Also, $\lim \sqrt{4 + x_n} = \sqrt{4 + L}$,

by the ALT (used twice — once for addition, once for $\sqrt{\ }$).

Thus $\quad x_{n+1} = \sqrt{4 + x_n}$,

we can take the limit of both sides,

and $\quad L = \sqrt{4 + L}$

$$\Rightarrow L^2 = 4 + L \Rightarrow L^2 - L - 4 = 0$$

$$\Rightarrow L = \frac{1 \pm \sqrt{1 + 16}}{2}$$

$$L = \frac{1 \pm \sqrt{17}}{2}.$$

Since $x_n \geq 1 \ \forall n \in \mathbb{N}$,

$L \geq 1$ by the OLT.

$\Rightarrow \boxed{L = \frac{1+\sqrt{17}}{2}}.$

$\approx 2.56155\ldots$

Example. Let $x_1 = 10$. Find the limit.

for $n \geq 1$, let $x_{n+1} = x_n^2 - 2$.

Naive answer: let $x_n \to L$.

ALT $\Rightarrow L = L^2 - 2 \Rightarrow L^2 - L - 2 = 0$

$(L-2)(L+1) = 0$

$L = 2$ or $L = -1$.

But you can prove $x_n \geq 0 \ \forall n$.

$\Rightarrow \boxed{L=2}.$

Problem: $x_1 = 10$

$x_2 = 10^2 - 2 = 98$

$x_3 = 98^2 - 2 = \_\_\_\_$

$\hookrightarrow$ It is monotone, but not bounded.

So actually, no limit exists.

Tweaked example:

$$x_1 = 1.5$$
$$x_{n+1} = x_n^2 - 2$$

it may actually converge in this case.

(so it converges to 2 or −1?)

$x_1 = -1$

$x_{n+1} = x_n^2 - 2$

Definitely converges.

$(x_n) = (-1, -1, -1, \dots)$

Series

A series is a specific kind of sequence, written like $\sum_1^{\infty} a_k = \sum_{k=1}^{\infty} a_k$.

What this actual means is the sequence of partial sums. That is:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$

$$S_n = \sum_{k=1}^{n} a_k$$

really: $\sum a_k$ means (S_n).

We say a series converges if the corresponding partial sum sequence (S_n) converges.

We say $\sum_{k=1}^{\infty} a_k = L$ exactly when

$\lim S_n = L$.

ie. $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} S_n = L$.

Example: $\sum_{k=1}^{\infty} 2^{-k}$ is the sequence

$$S_1 = 2^{-1} = \frac{1}{2}$$
$$S_2 = 2^{-1}+2^{-2} = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$$
$$S_3 = 2^{-1}+2^{-2}+2^{-3} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8}$$

in fact we can show

$$S_k = \frac{2^k-1}{2^k}.$$

Pf: $S_1 = 2^{-1} = \frac{1}{2} = \frac{2^1-1}{2^1}$. ✓

Suppose $S_k = \dfrac{2^k-1}{2^k}$. Then $S_{k+1} = \overbrace{\dfrac{2^k-1}{2^k}}^{S_k} + 2^{-(k+1)}$

$$= \frac{2^k-1}{2^k}+\frac{1}{2^{k+1}} = \frac{2(2^k-1)+1}{2^{k+1}} = \frac{2^{k+1}-2+1}{2^{k+1}} = \frac{2^{k+1}-1}{2^{k+1}}. ✓$$

$\therefore$ By induction, $S_k = \frac{2^k - 1}{2^k}$ $\forall$ k.

Lemma: If $|r| < 1$, then (r^n) converges to 0.

Pf. Scratch
Enough to show $|r|^n \to 0$
then use squeeze.
$|r|^{n+1} \leq |r|^n$ monotone decreasing
$0 \leq |r|^n \leq 1 \to$ bdd.
converges to L.
$\lim |r|^n = \lim |r|^{n+1} = |r| \lim |r|^n$
$\Rightarrow L = |r| L \Rightarrow L = 0.$

Pf: Observe that since $0 \leq |r| < 1$, then

$$0 \leq |r|^n < 1 \xleftarrow{n \text{ times}} \begin{array}{c} 0 \leq A < B \\ 0 \leq C < D \\ \Downarrow \\ AC < BD \end{array}$$

So the sequence $(|r|^n)$ is bounded.

also, $\forall n \in \mathbb{N}$, $|r|^{n+1} = |r|\,|r|^n < |r|^n$, so $(|r|^n)$ is decreasing.

By MCT, $(|r|^n)$ converges to a limit L.

But then $\lim |r|^{n+1} = \lim |r|^n$

$\Rightarrow |r| \lim |r|^n = \lim |r|^n$ by ALT

$\Rightarrow |r| L = L \Rightarrow (|r| - 1) L = 0$

$\Rightarrow L = 0.$

$\therefore \lim |r|^n = 0$

Then, since $-|r|^n \le r^n \le |r|^n$

and $\lim -|r|^n = 0 = \lim |r|^n$, by

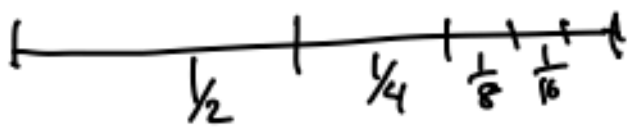
Squeeze Thm, $\lim r^n = 0$. $\boxtimes$

Our series $\sum 2^{-k}$ has a partial sum

$$S_k = \frac{2^k - 1}{2^k} = 1 - \left(\frac{1}{2}\right)^k$$

$\lim S_k = 1 - \lim \left(\frac{1}{2}\right)^k = 1$. Go frogs.

$$\Rightarrow \sum_{k=1}^{\infty} 2^{-k} = 1.$$



More general series:

<u>Geometric Series</u>

$$\sum_{k=0}^{\infty} a \cdot r^k$$

a is a fixed real #.

$r \in \mathbb{R}$.

<u>Thm</u>: $\sum_{k=0}^{\infty} a r^k$ converges $\iff |r| < 1$.

in fact $S_k = a\,\frac{1 - r^{k+1}}{1 - r}$, $\lim S_k = \frac{a}{1-r}$

(in those cases) $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

for $|r|<1$.

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for $0<r<1$, $a>0$

This (S_k) is an increased, bounded sequence $\Rightarrow$ (by MCT) it converges.

Idea of Proof (you can use induction to make this rigorous.):

$$S_k = a + ar + ar^2 + ar^3 + \dots ar^k$$

$$r S_k = \quad ar + ar^2 + ar^3 + \dots + ar^k + ar^{k+1}$$

subtract:

$$(1-r)S_k = a - ar^{k+1} \Rightarrow \boxed{S_k = \frac{a(1-r^{k+1})}{1-r.}}$$

$$\Rightarrow \lim S_k \underset{ALT}{=} \frac{a(1-0)}{1-r} = \frac{a}{1-r}.$$

$$\begin{pmatrix} \lim r^k = 0 \\ \lim r^{k+1} = 0 \end{pmatrix}$$
